An Approximate Wave Equation for an Axially Symmetric Periodic Waveguide

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Abstract—**The field problem of wave propagation in a waveguide of periodically varying section is investigated. An orthogonal curvilinear coordinate system is developed leading to a separable wave equation. As a result, the problem is reduced to solving Hill's Equation.**

The discussion is limited to the case of a waveguide with slowly varying radius but there is some expectation that useful results would be obtained, particularly for axial fields, without this restriction.

Introduction

A COMMON type of microwave periodic structure consists of a circular section wavequide loaded at regular intervals by metal irises. In analyzing the wave propogation in such a structure the usual procedure is to subdivide the structure into a number of regions, write down a set of wave functions for each, and endeavor to match field quantities across the boundaries of adjacent regions. A unified approach[1] is attempted in this paper, recognizing that the waveguide has a periodically changing radius.

The procedure, in essence, is to construct a system of curvilinear coordinates such that the boundary surface of the periodic waveguide coincides with a level surface of one of the coordinates. As a result, the question of wave propagation can be studied in terms of a single wave equation. The case of a waveguide with a slowly varying radius is considered here. An approximate wave equation is derived which is separable and, as a consequence, it turns out that the field problem can be reduced to finding the solution to Hill's Equation [5], [6].

Theory

Locate the cylindrical polar coordinate system (r, ϕ, z) so that the z-axis is the axis of the waveguide. The radius of the periodic structure to be investigated may be written as

$$\bar{r} = \bar{u}_1\left[1 + bf\left(\frac{2\pi z}{p}\right)\right] \tag{1}$$

where $f(2\pi z/p)$ is a periodic function with period p.

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[1] An example of a unified approach to the problem of a periodic waveguide was given by Cullen [1] who studied the sinuous waveguide by perturbing the axis of a regular waveguide. A number of authors [2]–[4] have treated the problem of the tapered waveguide by developing a suitable curvilinear coordinate system. Their approach has been to set the product of the slopes of two orthogonal coordinates equal to −1 and, knowing one coordinate, to construct an integral expression for the other coordinate.

Also,

$$0 \leq \bar{u}_1, \qquad 0 < b < 1, \qquad \left|f\left(\frac{2\pi z}{p}\right)\right| \leq 1.$$

A new system of curvilinear coordinates (u_1, u_2, u_3) is chosen such that at the waveguide wall, u_1 is constant and equal to $\bar{u}_1$. Also, u_3 is regarded as a function that is perturbed from z. Consequently, the new system to be introduced is

$$\begin{aligned} u_1 &= \frac{r}{1 + bf\left(\frac{2\pi z}{p}\right)} \\ u_2 &= \phi \\ u_3 &= z + \Delta(r, z) \end{aligned} \tag{2}$$

where Δ is to be determined.

In the interests of simplicity it is desirable that the $u_i (i=1, 2, 3)$ coordinates are orthogonal and lead to a separable wave equation. These goals can be achieved to a close approximation if, as will be seen, the radius of the waveguide varies slowly along the length of the structure.

In seeking a definition of Δ the orthogonality requirement will be considered first. To do this it is helpful to express the old coordinates explicitly in terms of the new. The fact that Δ is a small quantity justifies the neglect of higher order terms in Taylor expansions in Δ, thereby considerably simplifying the problem.

If $f(2\pi z/p)$ is expanded in a Taylor series about u_3,

$$f\left(\frac{2\pi z}{p}\right) = f\left(\frac{2\pi u_3}{p}\right) - \frac{2\pi}{p} f'\left(\frac{2\pi u_3}{p}\right)\Delta(r, z) + \cdots$$

where

$$f'\left(\frac{2\pi u_3}{p}\right) = \frac{df\left(\frac{2\pi u_3}{p}\right)}{d\left(\frac{2\pi u_3}{p}\right)}.$$

Therefore, when only lowest order terms are retained

$$r \cong u_1\left[1 + bf\left(\frac{2\pi u_3}{p}\right)\right] \tag{3}$$

$$\Delta(r, z) \cong \Delta\left[u_1\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right), u_3\right]. \tag{4}$$

From (2)–(4) the old coordinates can be expressed in terms of the new coordinates approximately as

$$
\begin{aligned}
r &= u_1\left[1 + bf\left(\frac{2\pi u_3}{p}\right)\right] \\
\phi &= u_2 \\
z &= u_3 - \Delta\left[u_1\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right), u_3\right].
\end{aligned} \tag{5}
$$

In these approximations it is assumed that $f(2\pi u_3/p)$ is sufficiently slowly varying to ensure that the perturbation of r in going from z to u_3 is small and so

$$\left|\frac{2\pi b}{p} f'\left(\frac{2\pi u_3}{p}\right)\Delta(r, z)\right| \ll 1 + bf\left(\frac{2\pi u_3}{p}\right). \tag{6}$$

Also, it is taken that $\Delta(r, z)$ varies slowly with z. From the Taylor series expansion in r and z of Δ and by using the approximation for the perturbation in r of

$$\Delta r \cong -\frac{2\pi u_1}{p} bf'\left(\frac{2\pi u_3}{p}\right)\Delta(r, z)$$

it is seen that (4) is satisfied if

$$\left|\frac{\partial \Delta\left[u_1\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right), u_3\right]}{\partial u_3}\right| \ll 1. \tag{7}$$

If $\boldsymbol{a}_i (i=1, 2, 3)$ is the unitary vector in the u_i direction,

$$\boldsymbol{a}_i = h_i \boldsymbol{i}_i$$

where h_i is the Lamé coefficient and $\boldsymbol{i}_i$ is the unit vector. From (5), [7]

$$
\begin{aligned}
h_1 &\cong \sqrt{\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right)^2 + \left(\frac{\partial \Delta}{\partial u_1}\right)^2} \\
h_2 &\cong u_1\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right) \\
h_3 &\cong \sqrt{\left(1 - \frac{\partial \Delta}{\partial u_3}\right)^2 + \left(\frac{2\pi u_1}{p} bf'\left(\frac{2\pi u_3}{p}\right)\right)^2}
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
\boldsymbol{i}_1 &= \frac{\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right)\boldsymbol{i}_r - \frac{\partial \Delta}{\partial u_3}\boldsymbol{i}_z}{\sqrt{\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right)^2 + \left(\frac{\partial \Delta}{\partial u_1}\right)^2}} \\
\boldsymbol{i}_2 &= \boldsymbol{i}_\phi \\
\boldsymbol{i}_3 &= \frac{\frac{2\pi u_1}{p} bf'\left(\frac{2\pi u_3}{p}\right)\boldsymbol{i}_r + \left(1 - \frac{\partial \Delta}{\partial u_3}\right)\boldsymbol{i}_z}{\sqrt{\left(1 - \frac{\partial \Delta}{\partial u_3}\right)^2 + \left(\frac{2\pi u_1}{p} bf'\left(\frac{2\pi u_3}{p}\right)\right)^2}}
\end{aligned} \tag{9}
$$

where $\boldsymbol{i}_r$, $\boldsymbol{i}_\phi$, $\boldsymbol{i}_z$ are the unit vectors in a cylindrical coordinate system. Therefore, $\boldsymbol{i}_2$ is orthogonal to $\boldsymbol{i}_1$ and $\boldsymbol{i}_3$ and it is readily shown that $\boldsymbol{i}_1$ and $\boldsymbol{i}_3$ are orthogonal when

$$\frac{\partial \Delta}{\partial u_1} \cong \frac{2\pi u_1}{p} bf'\left(\frac{2\pi u_3}{p}\right)\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right). \tag{10}$$

Since along the axis of the waveguide Δ must be zero, integrating (10) gives

$$\Delta = \frac{1}{2}\left(\frac{2\pi}{p}\right) u_1^2 bf'\left(\frac{2\pi u_3}{p}\right)\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right) \tag{11}$$

and this expression is taken as the definition of Δ.

To ensure that the wave equation is separable, the restriction

$$\left(\frac{2\pi u_1}{p} bf'\left(\frac{2\pi u_3}{p}\right)\right)^2 \ll 1 \tag{12}$$

is made. Consequently, from (8) along with (7)

$$
\begin{aligned}
h_1 &\cong 1 + bf\left(\frac{2\pi u_3}{p}\right) \\
h_2 &\cong u_1\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right) \\
h_3 &\cong 1
\end{aligned} \tag{13}
$$

and, as will be shown, for these Lamé coefficients a separable solution exists.

Through the use of (11) and (12) it can be seen that (6) is satisfied. Now, (11) is differentiated with respect to u_3 and the resulting expression, when substituted into (7), gives the condition

$$\left|\left(\frac{2\pi u_1}{p}\right)^2 bf''\left(\frac{2\pi u_3}{p}\right)\left[1 + bf\left(\frac{2\pi u_3}{p}\right)\right]\right| \ll 1. \tag{14}$$

The conditions imposed on the waveguide parameters are not as restricting as they might appear. For example, a check will show that the parameters, given in (1), of a structure with a radius varying as

$$r = 0.36(1 + 0.5 \cos z)$$

easily satisfy the conditions. Such a waveguide has a noteworthy amount of loading, since b is a measure of the loading.

Of the many possible wave types which may exist in a structure, the most interesting from a practical point of view are those in which either the electric or magnetic intensity has no component in the axial direction. Choosing the latter, it may be shown [8] that $H_1 \equiv 0$, $E_2 \equiv 0$, $H_3 \equiv 0$ and there is no u_2 dependence in any field quantity (axially symmetric fields). Therefore, from Maxwell's equations, if E_1 and E_2 are eliminated

$$u_1 \frac{\partial}{\partial u_1}\left[\frac{1}{u_1}\frac{\partial}{\partial u_1}(h_2 H_2^*)\right] + h_1^2\left[\frac{\partial^2}{\partial u_3^2}(h_2 H_2^*) + \omega^2 \mu_1 \epsilon_1 h_2 H_2^*\right] = 0 \tag{15}$$

where μ_1 and ϵ_1 are constant, the h_i's are given in (13) and H_2^* is an approximation of H_2 since the h_i's are approximate. Equation (15) is separable and $h_2H_2^*$ can be expressed (with time dependence suppressed) as

$$h_2H_2^* = R(u_1)T(u_3). \tag{16}$$

If (16) is substituted into (15), the result is

$$u_1\frac{d}{du_1}\left[\frac{1}{u_1}\frac{dR}{du_1}\right] + K^2R = 0 \tag{17}$$

$$\frac{d^2T}{du_3^2} + \left[\omega^2\mu_1\epsilon_1 - \left(\frac{K}{h_1}\right)^2\right]T = 0 \tag{18}$$

where K is a separation constant.

The solution to Bessel's equation (17) is $R = u_1J_1(Ku_1)$. Since at the wall of the waveguide $E_3 = 0$, the boundary condition to be fulfilled is $J_0(K\bar{u}_1) = 0$.

If (13) is used to eliminate h_1, from (18), then

$$\frac{d^2T}{du_3^2} + \left[\omega^2\mu_1\epsilon_1 - \frac{K^2}{\left(1 + bf\left(\frac{2\pi u_3}{p}\right)\right)^2}\right]T = 0. \tag{19}$$

Equation (19) is commonly known as Hill's equation and has been extensively treated in the literature. For example, a general solution method is discussed by Whittaker and Watson [5] and also by Brillouin [6] in which T is expanded in the series

$$T = \sum_{n=-\infty}^{\infty} a_n e^{-j(x+2n\pi/p)u_3}.$$

An approximate solution may be found by truncating the series. Solutions using perturbation theory have been given by Brillouin [9] and McLachlan [10].

Once T is determined, from Maxwell's equations and (16), the field is known and

$$E_1 \cong \frac{jJ_1(Ku_1)}{\omega\epsilon_1\left[1 + bf\left(\frac{2\pi u_3}{p}\right)\right]}\frac{dT}{du_3}$$

$$H_2 \cong \frac{J_1(Ku_1)T(u_3)}{1 + bf\left(\frac{2\pi u_3}{p}\right)}$$

$$E_3 \cong -j\frac{KJ_0(Ku_1)T(u_3)}{\omega\epsilon_1\left[1 + bf\left(\frac{2\pi u_3}{p}\right)\right]}. \tag{20}$$

The field component, E_z, can be determined from

$$E_z = \boldsymbol{i}_z\cdot\boldsymbol{E} = \boldsymbol{i}_z\cdot\boldsymbol{i}_1E_1 + \boldsymbol{i}_z\cdot\boldsymbol{i}_3E_3.$$

Through the use of (9) and (10)

$$E_z \cong -\frac{2\pi u_1}{p}bf'\left(\frac{2\pi u_3}{p}\right)E_1 + E_3.$$

Along the axis of the waveguide $\Delta = 0$, $z = u_3$, and hence

$$E_z \cong -j\frac{KT(z)}{\omega\epsilon_1\left[1 + bf\left(\frac{2\pi z}{p}\right)\right]}.$$

For the limiting case in which $b\to 0$, $u_1\to r$ and $u_3\to z$, (20) becomes

$$E_r = j\frac{J_1(Kr)}{\omega\epsilon_1}\frac{dT}{dz}$$

$$H_\phi = J_1(Kr)T(z)$$

$$E_z = -j\frac{KJ_0(Kr)}{\omega\epsilon_1}T(z) \tag{21}$$

where T satisfies the differential equation

$$\frac{d^2T}{dz^2} + (\omega^2\mu_1\epsilon_1 - K^2)T = 0.$$

It can be recognized that (21) is an E-wave solution in a uniform circular-section waveguide. Hence, in the limit (20) is in agreement with the known solution.

Discussion

The theory just developed gives a relatively simple field solution method for an axially symmetric periodic structure with a slowly varying radius.

For a structure in which the radius of the walls is not slowly varying, the applicability of the present development has not been examined. However, as a point of speculation, it might be quite meaningful to employ the expressions in (20) in order to obtain the field at least on the axis of the structure, and as a consequence the method should prove useful in the design of specific structures for beam couplers.

The method could also be adapted to include structures with the period a function of z. No new difficulty would be experienced in establishing a suitable coordinate system and a separable wave equation. It would, however, be necessary to solve a second-order differential equation with coefficients of variable period.

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